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LETTR TO THE EDITOR

Spherical Raman–Nath equation for gyrotron radiation

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Abstract. Based on the structure of the energy levels of a relativistic electron moving in a uniform magnetic field first derived by Schneider, a more rigorous quantum theory of gyrotron radiation is developed. As the result, a difference-differential equation is obtained to describe the evolution of gyrotron radiation. This equation is of the form of the spherical Raman–Nath equation, well known in the quantum theory of free-electron lasers. Therefore, all the mathematical techniques developed for solving the spherical Raman–Nath equation can be used immediately to solve the problem of gyrotron radiation.

Gyrotron radiation was first advocated by Schneider (1959) in a purely quantum mechanical context as the 'stimulated emission of radiation by relativistic electrons in a magnetic field' when he solved the relativistic Schrödinger equation for an electron in a uniform magnetic field. However, the subsequent theoretical developments have been carried out completely based on classical relativistic dynamics and plasma waves. Recently, Ho and Granatstein (1986) have adopted the viewpoint of quantum electronics for their analytical study of the gyrotron; it is viewed as the coherent radiation of induced linear electric dipole moments. In this letter, we will reduce a gyrotron to its bare essentials and focus our attention on an individual electron as it cascades down the energy-level ladder and the accompanying radiation.

We consider a relativistic electron in a static and uniform magnetic field of magnitude B pointing along the z axis. It is well known that the relativistic wavefunction in the xy plane can be expressed in terms of the generalised Laguerre polynomials as (Sokolov and Ternov 1986)

$$|j, l\rangle \equiv \Psi_{j,l}(s, \phi) = \sqrt{\frac{(j-l)!}{2\pi(j!)}}\,\mathrm{e}^{il\phi - s/2} s^{l/2} L^{l}_{j-l}(s) \tag{1}$$

where

$$s \equiv \frac{eB}{2\hbar}(x^2+y^2) \tag{2}$$

where $\hbar$ is Planck's constant, ϕ is the azimuth angle, and $L^{l}_{j-1}(s)$ is the generalised Laguerre polynomial. The corresponding eigenenergy is

$$E_j = m_0c^2[1+(2j+1)\hbar\omega_0/m_0c^2]^{1/2} \tag{3}$$

where

$$\omega_0 \equiv eB/m_0 \tag{4}$$

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is the cyclotron frequency. It should be noticed that the eigenenergy does not depend on l.

The binomial expansion of (3) up to the second order of j gives

$$E_j \approx E_0 + j\hbar\omega - j^2\hbar\varepsilon \tag{5}$$

where $E_0 \equiv m_0c^2(1+\delta/2-\delta^2/8)$, $\omega \equiv \omega_0(1-\delta/2)$, and $\hbar\varepsilon \equiv m_0c^2\delta^2/2$ with $\delta \equiv \hbar\omega_0/m_0c^2$. Since E_0 is a constant, it can be dropped from now on simply by shifting the energy scale.

Consider the interaction of the electron with a radiation field. Let an arbitrary quantum state of the system at time t be expressed as

$$|\psi(t)\rangle = \sum_{j=0}^{\infty}\sum_{l=-\infty}^{j}\sum_{n=0}^{\infty} C_{j,l,n}(t)\,\mathrm{e}^{-\mathrm{i}(j+n)\omega t}|j, l\rangle|n\rangle \tag{6}$$

where $|n\rangle$ is the photon-number state or Fock state and the $C_{j,l,n}(t)$ are slowly varying probability amplitudes. Let the Hamiltonian of the system be written as

$$H = H_0 + H_{\mathrm{int}} \tag{7}$$

with

$$H_0|j, l\rangle|n\rangle = [(n+j)\hbar\omega - j^2\hbar\varepsilon]|j, l\rangle|n\rangle \tag{8}$$

and

$$H_{\mathrm{int}} = -e\boldsymbol{E}\cdot\boldsymbol{r} = \mathrm{i}e\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}(a^\dagger - a)y \tag{9}$$

where $\boldsymbol{E}$ is the electric field of the radiation assumed to be polarised along the y axis, $a^\dagger$ and a are the creation and the annihilation operators, respectively, ε_0 is the permittivity of the vacuum, and V is the quantisation volume.

Using the properties of generalised Laguerre polynomials, we can calculate matrix elements of y to be as follows:

$$\langle j, l|y|j', l'\rangle = \mathrm{i}\sqrt{\frac{\hbar}{2eB}}\begin{cases} -\sqrt{j} & l'=l-1, j'=j-1 \\ \sqrt{j-l+1} & l'=l-1, j'=j \\ -\sqrt{j-l} & l'=l+1, j'=j \\ \sqrt{j+1} & l'=l+1, j'=j+1 \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Using (7)-(10) in the Schrödinger equation

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = H|\psi(t)\rangle \tag{11}$$

multiplying by $\langle n|\langle j, l|$ from the left and using the orthogonality properties of the eigenstates, we obtain

$$\begin{aligned}\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}C_{j,l,n} = &-\varepsilon j^2 C_{j,l,n} - \Lambda\sqrt{(j+1)(n+1)}\,C_{j+1,l+1,n-1}\,\mathrm{e}^{-2\mathrm{i}\omega t} \\ &+\Lambda\sqrt{(j-l)(n+1)}\,C_{j,l+1,n+1}\,\mathrm{e}^{-\mathrm{i}\omega t} + \Lambda\sqrt{j(n+1)}\,C_{j-1,l-1,n+1} \\ &-\Lambda\sqrt{(j-l+1)(n+1)}\,C_{j,l-1,n+1}\,\mathrm{e}^{-\mathrm{i}\omega t} \\ &+\Lambda\sqrt{(j+1)n}\,C_{j+1,l+1,n-1} - \Lambda\sqrt{(j-1)n}\,C_{j,l+1,n-1}\,\mathrm{e}^{\mathrm{i}\omega t} \\ &-\Lambda\sqrt{jn}\,C_{j-1,l-1,n-1}\,\mathrm{e}^{2\mathrm{i}\omega t} + \Lambda\sqrt{(j-l+1)n}\,C_{j,l-1,n-1}\,\mathrm{e}^{\mathrm{i}\omega t}\end{aligned} \tag{12}$$

where $\Lambda \equiv \sqrt{e\omega/4\varepsilon_0 BV}$ is the coupling constant.

We now invoke the rotating-wave approximation to drop those terms with rapidly oscillating factors in (12) to obtain

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}C_{j,l,n}(t)=-\varepsilon j^2C_{j,l,n}(t)+\Lambda\sqrt{j(n+1)}C_{j-1,l-1,n+1}(t)+\Lambda\sqrt{(j+1)n}C_{j+1,l-1,n-1}(t). \tag{13}$$

It is obvious that l is an irrelevant quantum number; so we let

$$C_{j,n}(t)\equiv\sum_{l=-\infty}^{j}C_{j,l,n}(t). \tag{14}$$

Then (13) is reduced to

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}C_{j,n}(t)=-\varepsilon j^2C_{j,n}(t)+\Lambda\sqrt{j(n+1)}C_{j-1,n+1}(t)+\Lambda\sqrt{(j+1)n}C_{j+1,n-1}(t). \tag{15}$$

It is also obvious that $j+n=N$ is an invariant in the evolution governed by (15), which is also required by conservation of energy. Therefore, only one index is sufficient to identify the probability amplitude $C_j(t)\equiv C_{j,n}(t)$. So the equation can be further reduced to

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}C_j(t)=-\varepsilon j^2C(t)+\Lambda\sqrt{j(N-j+1)}C_{j-1}(t)+\Lambda\sqrt{(j+1)(N-j)}C_{j+1}(t). \tag{16}$$

This difference-differential equation is of the type of spherical Raman–Nath (RN) equation well known in the quantum theory of free-electron lasers (Ciocci *et al* 1986). The original RN equation was derived to describe light diffraction by ultrasound (Raman and Nath 1937). The unique feature of this type of equation is the existence of a term proportional to j^2 which indicates the characteristics of a nonlinear problem. This is the very reason that the RN equation has defied any exact solution in its long history of existence. However, becaususe these types of equations appear in a large number of physical phenomena, as pointed out by Bosco and Dattoli (1983), a vast collection of mathematical tools have been developed to solve them (Bosco and Dattoli 1983, Bosco *et al* 1984, Lee 1985, 1987, 1988, Ciocci *et al* 1986, Carusotto 1989). Therefore, we can utilise this great wealth of mathematical techniques immediately to study the problem of gyrotron radiation quantum mechanically.

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